TEICHMÜLLER CONTRACTION IN THE TEICHMÜLLER SPACE OF A CLOSED SET IN THE SPHERE

BY

SUDEB MITRA

Department of Mathematics, University of Connecticut
Storrs, CT 06269, USA
e-mail: mitra@math.uconn.edu

ABSTRACT

We generalize the principle of Teichmüller contraction and deduce the Hamilton–Krushkaĺ condition for extremal quasiconformal mappings in the Teichmüller space of a closed set in the Riemann sphere.

1. Introduction

Throughout this paper, we shall assume that E is a closed subset of the Riemann sphere, and that 0, 1, and ∞ belong to E. The Teichmüller space of E, which we denote by $T(E)$, was first studied by G. Lieb in his doctoral dissertation [11]. He proved that $T(E)$ is a complex Banach manifold and gave some applications of $T(E)$ to the theory of holomorphic motions. In [12], it was shown that $T(E)$ is a universal parameter space for holomorphic motions of the set E over a simply connected complex Banach manifold. Further properties of $T(E)$ and applications to holomorphic motions have been reported in the recent papers [3], [4], [6] and [12].

In this paper we study the Teichmüller metric on $T(E)$. We prove the principle of Teichmüller contraction for $T(E)$ and use it to obtain the Hamilton–Krushkaĺ condition. N. Lakic has an independent unpublished proof of Teichmüller contraction for $T(E)$, based on extending the Reich–Strebel inequalities to the $T(E)$ setting. We base our proof on an approximation technique that was central in

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[12]. The idea is to apply the classical principle of Teichmüller contraction to the Teichmüller space of the sphere punctured at finitely many points of E. The present paper is a sequel to [12].

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2. The Teichmüller metric on $T(E)$

In this section we summarize some basic facts about the Teichmüller space $T(E)$. We refer the reader to [6] for a more detailed discussion.

2.1. BASIC DEFINITIONS. Recall that a homeomorphism of $\widehat{\mathbb{C}}$ is called **normalized** if it fixes the points 0, 1, and ∞. Two normalized quasiconformal self-mappings f and g of $\widehat{\mathbb{C}}$ are called E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The **Teichmüller space** $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$. The **basepoint** of $T(E)$ is the E-equivalence class of the identity map.

Let $M(\mathbb{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\widehat{\mathbb{C}}$ onto itself, so we shall often refer to the elements of $M(\mathbb{C})$ as Beltrami coefficients. The basepoint of $M(\mathbb{C})$ is the zero function.

We define the quotient map $P_E : M(\mathbb{C}) \to T(E)$ by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ. Obviously P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$. In [11] Lieb proved that $T(E)$ is a complex Banach manifold such that the map P_E from $M(\mathbb{C})$ to $T(E)$ is a holomorphic split submersion (also see [6]).

The Teichmüller distance $d_M(\mu, \nu)$ between μ and ν on $M(\mathbb{C})$ is defined by

$$d_M(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \bar{\mu}\nu} \right\|_\infty .$$

The **Teichmüller metric** on $T(E)$ is the quotient metric

$$d_{T(E)}(s, t) = \inf\{d_M(\mu, \nu) : \mu \text{ and } \nu \text{ in } M(\mathbb{C}), P_E(\mu) = s \text{ and } P_E(\nu) = t\}$$

for all s and t in $T(E)$.

2.2. FORGETFUL MAPS. If E is a subset of the closed set $\tilde{E}$ and μ is in $M(\mathbb{C})$, then the $\tilde{E}$-equivalence class of w^{μ} is contained in the E-equivalence class of w^{μ}. Therefore, there is a well-defined 'forgetful map' $p_{\tilde{E},E}$ from $T(\tilde{E})$ to $T(E)$ such that $P_E = p_{\tilde{E},E} \circ P_{\tilde{E}}$. It is easy to see that this forgetful map is a basepoint preserving holomorphic split submersion. It is also weakly distance decreasing with respect to the Teichmüller metrics.

3. The main results

In this section 0_M and 0_T denote the basepoints of $M(\mathbb{C})$ and $T(E)$ respectively.

Definition 3.1: A Beltrami coefficient μ in $M(\mathbb{C})$ is extremal in its E-equivalence class if $d_{T(E)}(0_T, P_E(\mu)) = d_M(0_M, \mu)$.

Definition 3.2: We denote by $A(E)$ the closed subspace of $L^1(\mathbb{C})$ consisting of the functions f in $L^1(\mathbb{C})$ whose restriction to the complement $E^c = \mathbb{C} \setminus E$ of E is holomorphic.

Definition 3.3: Let $\mu \in L^{\infty}(\mathbb{C})$. The **Hamilton–Krushkaĺ norm** of μ relative to $T(E)$ is defined by:

$$\|\mu\|_{HK(E)} = \sup_{\|\phi\|=1} \left| \int\int_{\mathbb{C}} \mu\phi dxdy \right|, \quad \phi \in A(E).$$

In other words, $\|\mu\|_{HK(E)}$ is the norm of the linear functional

$$\ell_{\mu}(\phi) = \int\int_{\mathbb{C}} \mu\phi dxdy \quad \text{on } A(E).$$

The following theorem is the principle of Teichmüller contraction for $T(E)$.

THEOREM 3.1: *Let $\mu \in M(\mathbb{C})$ be given with $\|\mu\|_{\infty} > 0$ and let $P_E(\mu) = \tau$ in $T(E)$. Then we have the following:*

Given $\epsilon > 0$, there exists a $\delta > 0$ depending only on ϵ and $\|\mu\|_{\infty}$ such that

$$\|\mu\|_{HK(E)} \leq (1-\delta)\|\mu\|_{\infty} \quad \text{if } d_{T(E)}(0_T, \tau) \leq d_M(0_M, \mu) - \epsilon. \tag{3.1}$$

Given $\delta > 0$, there exists an $\epsilon > 0$ depending only on δ and $\|\mu\|_{\infty}$ such that

$$d_{T(E)}(0_T, \tau) \leq d_M(0_M, \mu) - \epsilon \quad \text{if } \|\mu\|_{HK(E)} \leq (1-\delta)\|\mu\|_{\infty}. \tag{3.2}$$

As a corollary we obtain the Hamilton–Krushkaĺ condition for extremality in $T(E)$:

COROLLARY 3.2: *A Beltrami coefficient μ is extremal in its E-equivalence class if and only if $\|\mu\|_\infty = \|\mu\|_{HK(E)}$.*

Remark: Our proof of Theorem 3.1 yields explicit estimates for δ and ϵ in (3.1) and (3.2) respectively. See the remark in §6.

4. The finite case

Let E be a finite set. Its complement $E^c = \Omega$ is the Riemann sphere with punctures at the points of E. Since $T(E)$ and the classical Teichmüller space $Teich(\Omega)$ are quotients of $M(\mathbb{C})$ by the same equivalence relation, $T(E)$ can be naturally identified with $Teich(\Omega)$ (see [12]). Under this identification $d_{T(E)}$ becomes the (classical) Teichmüller metric for $Teich(\Omega)$ (see [13] for the standard definitions).

In addition, when E is finite the Hamilton–Krushkaĭ norm of μ relative to $T(E)$ is simply the norm of the linear functional that μ induces on the Banach space of integrable holomorphic functions on Ω. Therefore in the finite case Theorem 3.1 and Corollary 3.2 are the classical principle of Teichmüller contraction and Hamilton–Krushkaĭ condition for $Teich(\Omega)$ (see [8] and [7] respectively; also see [9]).

5. Approximation by finite subsets

Let E be infinite and let $E_1, E_2, \ldots, E_n, \ldots$ be a sequence of finite subsets of E such that $\{0, 1, \infty\} \subset E_1 \subset \cdots \subset E_n \subset \cdots$ and $\bigcup_{n=1}^{\infty} E_n$ is dense in E.

Let 0 be the basepoint of $T(E)$, and for each $n \geq 1$, let π_n be the forgetful map p_{E,E_n} from $T(E)$ to $T(E_n)$ (see §2.2). For any τ in $T(E)$ and $n \geq 1$ let $\tau_n = \pi_n(\tau)$. In particular, $0_n = \pi_n(0)$ is the basepoint of $T(E_n)$ for all $n \geq 1$.

Since forgetful maps are weakly distance decreasing it is easy to see that

$$(5.1) \qquad d_{T(E_n)}(0_n, \tau_n) \leq d_{T(E_{n+1})}(0_{n+1}, \tau_{n+1}) \leq d_{T(E)}(0, \tau)$$

for all τ in $T(E)$ and $n \geq 1$. Our proof of Theorem 3.1 depends crucially on the following stronger result, which is proved in [12].

LEMMA 5.1: *For each τ in $T(E)$ the increasing sequence $\{d_{T(E_n)}(0_n, \tau_n)\}$ converges to $d_{T(E)}(0, \tau)$.*

We shall also need the following lemma, which is an analogue of Lemma 5.1 for the spaces $A(E_n)$ and $A(E)$.

LEMMA 5.2: *Let the infinite closed set E and the finite subsets E_n, $n \geq 1$, be as above, and let μ belong to $L^\infty(\mathbb{C})$. The sequence $\{\|\mu\|_{HK(E_n)}\}$ is increasing and converges to $\|\mu\|_{HK(E)}$.*

Proof: Since $E_n \subset E_{n+1} \subset E$ for $n \geq 1$, it is obvious from Definition 3.2 that $A(E_n) \subset A(E_{n+1}) \subset A(E)$ for all $n \geq 1$. Definition 3.3 therefore implies that the sequence $\{\|\mu\|_{HK(E_n)}\}$ is increasing and is bounded above by $\|\mu\|_{HK(E)}$. To prove that the sequence converges to $\|\mu\|_{HK(E)}$ it suffices to show that the union of the spaces $A(E_n)$ is dense in $A(E)$.

Since each E_n is a finite set, each $A(E_n)$ is the finite dimensional space of rational functions in $L^1(\mathbb{C})$ whose poles all belong to E_n. Therefore the union of the spaces $A(E_n)$ is the space of rational functions in $L^1(\mathbb{C})$ whose poles all belong to the dense subset $\bigcup_n E_n$ of E. That space of rational functions is dense in $A(E)$ by a theorem of Lakic (see the proof of Corollary 7 in [10] or Corollary 2.2 in [2]). ∎

6. Proofs of the main results

Proof of Theorem 3.1: As we saw in §4, the fact that each E_n is finite implies the following:

(i) Given $\epsilon > 0$, there exists a $\delta > 0$, depending only on ϵ and $\|\mu\|_\infty$ such that for every $n \geq 1$:

$$\|\mu\|_{HK(E_n)} \leq (1-\delta)\|\mu\|_\infty \quad \text{if } d_{T(E_n)}(0_n, \tau_n) \leq d_M(0_M, \mu) - \epsilon. \tag{6.1}$$

(ii) Given $\delta > 0$, there exists an $\epsilon > 0$, depending only on δ and $\|\mu\|_\infty$ such that for every $n \geq 1$:

$$d_{T(E_n)}(0_n, \tau_n) \leq d_M(0_M, \mu) - \epsilon \quad \text{if } \|\mu\|_{HK(E_n)} \leq (1-\delta)\|\mu\|_\infty. \tag{6.2}$$

Suppose that $\epsilon > 0$ is given and $d_{T(E)}(0, \tau) \leq d_M(0_M, \mu) - \epsilon$. By (5.1) we have $d_{T(E_n)}(0_n, \tau_n) \leq d_M(0_M, \mu) - \epsilon$ for each $n \geq 1$. Therefore, by (6.1) there exists a $\delta > 0$, depending only on ϵ and $\|\mu\|_\infty$, such that for each $n \geq 1$, we have $\|\mu\|_{HK(E_n)} \leq (1-\delta)\|\mu\|_\infty$. Lemma 5.2 immediately gives $\|\mu\|_{HK(E)} \leq (1-\delta)\|\mu\|_\infty$ and this proves (3.1).

Next, suppose that $\delta > 0$ is given and $\|\mu\|_{HK(E)} \leq (1-\delta)\|\mu\|_\infty$. By Lemma 5.2, we have $\|\mu\|_{HK(E_n)} \leq (1-\delta)\|\mu\|_\infty$ for each $n \geq 1$. By (6.2) there exists an $\epsilon > 0$, depending only on δ and $\|\mu\|_\infty$ such that $d_{T(E_n)}(0_n, \tau_n) \leq d_M(0_M, \mu) - \epsilon$ for each $n \geq 1$. It follows from Lemma 5.1, that $d_{T(E)}(0, \tau) \leq d_M(0_M, \mu) - \epsilon$ which proves (3.2). ∎

Remark: Theorem 10 in Chapter 4 of [9] yields explicit values for δ and ϵ in (6.1) and (6.2) respectively. The proof of Theorem 3.1 shows that these values can also be used in (3.1) and (3.2).

Writing $k = \|\mu\|_\infty$ we get

$$\delta = (1-k)^2(1-k^2)\frac{\tanh(\epsilon)}{8k}$$

for (3.1). Note that δ depends *only* on k and ϵ.

Similarly, for (3.2) (for $0 < \delta \leq 1$) we get

$$\tanh(\epsilon) = k(1-k)^2\delta$$

where ϵ depends *only* on k and δ. (For $\delta > 1$, (3.2) is vacuous.) The computations are straightforward and are left to the reader.

Proof of Corollary 3.2: The proof follows easily from Definition 3.1 and Theorem 3.1. In fact, if $\|\mu\|_\infty = \|\mu\|_{HK(E)}$ and μ is not extremal in its E-equivalence class, then there exists an $\epsilon > 0$ such that $d_{T(E)}(0, P_E(\mu)) < d_M(0_M, \mu) - \epsilon$. By Theorem 3.1, there exists a $\delta > 0$ such that $\|\mu\|_{HK(E)} \leq (1-\delta)\|\mu\|_\infty$ and we get a contradiction. The other direction is equally obvious. ∎

7. An open question

Earle, Gardiner and Lakic have defined the Asymptotic Teichmüller space $AT(X)$ when X is a Riemann surface and have shown that $AT(X)$ is a complex manifold (see [5]). The principle of Teichmüller contraction also holds for the Asymptotic Teichmüller space; see Chapter 14 of [9] for a proof. It would be interesting to define the Asymptotic Teichmüller space $AT(E)$ and study Teichmüller contraction in that setting.

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